THE MUTUAL AND INPUT IMPEDANCE OF STRIPS BETWEEN PARALLEL PLANES

William H. Hayt, Jr.
Purdue University
Lafayette, Indiana

Introduction

The input impedance of a thick dipole antenna between parallel planes, or the mutual impedance between two such elements, each carrying an assumed sinusoidal current distribution, is obtained by first considering filamentary dipoles. The longitudinal or tangential field intensity about a single dipole of length L between parallel planes, as indicated in Fig. 1, is therefore obtained by considering the infinite series of images in the two planes, from which the longitudinal field is simply written as an infinite summation of the fields due to the individual dipole elements. This series converges too slowly to be of any practical use, however, and a transformation of series must be applied[1], which results in the infinite series of exponential terms with imaginary arguments becoming a series of modified Bessel functions of zero order of the second kind. The final expression is given as Eq. (1) below and converges rapidly in most cases.[2]

$$E_x = -j\frac{120}{h} I_{mx} \sum_{n=1}^{\infty} \sin\frac{\pi n y'}{h} \sin\frac{\pi n y_1'}{h} \quad (1)$$

$$\left\{ K_o\left[\alpha_n\sqrt{\zeta^2+\beta^2(X-L/2)^2}\right] + K_o\left[\alpha_n\sqrt{\zeta^2+\beta^2(X+L/2)^2}\right] - 2\cos\frac{\beta L}{2} K_o\left[\alpha_n\sqrt{\zeta^2+\beta^2 X^2}\right] \right\}$$

where,

$$I_x = I_{mx}\sin\beta\,(L/2 - x)\,e^{jwt}, \quad L/2 \geq x \geq o$$

$$I_x = I_{mx}\sin\beta\,(L/2 + x)\,e^{jwt}, \quad -L/2 \leq x \leq o$$

$$\alpha_n = \sqrt{(\pi n/H)^2 - 1}, \quad H = \beta h, \quad \zeta = \beta Z \quad \beta = 2\pi/\lambda$$

$$y' = y - h/2 \quad , \quad y_1' = y_1 - h/2$$

Input Impedance

We are considering the filamentary dipole and therefore restrict the length of the dipole to resonant values to avoid infinite values of the input impedance. For simplicity, the dipole is also centered between the parallel planes. The induced emf method may be used to obtain the input impedance by evaluating Eq. (1) along the length of the antenna, a sinusoidal current distribution being assumed as before. After an obvious change of variable, we obtain, for the full dipole,

$$X_{in} = \frac{240}{H} \sum_{\substack{n=1\\ odd}}^{\infty} \int_0^{k\pi} \sin r \; K_o(\alpha_n r)\, dr \quad (2)$$

where,

$$L = k\,\lambda/2, \quad k = 1, 3, 5, \ldots, \text{(odd)}$$

If the guard plane separation is less than one-quarter of a wavelength, this equation may be approximated with an error of less than one-quarter of one per cent by extending the upper integration limit to infinity, giving,

$$X_{in} \doteq \frac{120}{\pi} \sum_{\substack{n=1\\ odd}}^{\infty} \frac{1}{M}\, \ell n\, \frac{1 + H/n\pi}{1 - H/n\pi} \quad (3)$$

or,

$$X_{in} \doteq \frac{120}{\pi}\left\{\frac{\pi H}{4} + \frac{\pi H^3}{144} + \sum_{\substack{n=1\\ odd}}^{\infty} \left[\frac{1}{M}\,\ell n\,\frac{1+H/n\pi}{1-H/n\pi} - \frac{2H}{\pi n^2} - \frac{2H^3}{3\pi^3 n^4}\right]\right\} \quad (3a)$$

Fig. 2 shows the input reactance of a half-wave dipole between parallel planes as a function of the plane separation. The values of input reactance for separations between one-quarter and one-half wavelength are obtained from Eq. (4) below, where the integral must be evaluated by numerical or graphical methods.

$$X_{in} \doteq \frac{240}{H}\int_0^{k\pi} \sin r \; K_o\left(r\sqrt{\frac{\pi^2}{H^2}-1}\right) dr - \frac{120}{\pi}\,\ell n\,\frac{1 + H/\pi}{1 - H/\pi} + \frac{120}{\pi}\sum_{\substack{n=1\\ odd}}^{\infty} \frac{1}{n}\,\ell n\,\frac{1+H/n\pi}{1-H/n\pi} \quad (4)$$

The impedance is a pure reactance for guard plane spearations of less than one-half wavelength, since no propagating mode can be established. At one-half wavelength separation, the input reactance becomes infinite as a resonance condition is established. For guard plane separations less than one-quarter wavelength, there is no significant difference in the input reactance for antennas having a full length of one-half, three halves, or other odd-half wavelengths.

As the antenna is moved closer to one of the guard planes, but remains parallel to both, the input reactance decreases in an approximately cosinusoidal manner, approaching zero as the dipole location coincides with that of the conducting plane.

Mutual Impedance

If we now consider the mutual impedance between two identical, resonant, dipole antennas, and for the moment allow both to be centered between the guard planes, the induced emf method leads to the following approximate result:

$$X_{12} \doteq \frac{240}{H} \sum_{\substack{n=1 \\ odd}}^{\infty} \sum_{\mu=0}^{\infty} (-1)^{\mu} \frac{2^{\mu}\mu!}{(2\mu+1)!} (y/d_n)^{\mu+1} K_{\mu+1}(\alpha_{\lambda} y) \quad (5)$$

which is accurate within 0.088 ohms for guard plane separations of less than one-quarter wavelength. This equation leads to the curves of Figs. 3 and 4, which differ only in the interchange of the parameter and independent variables.

An alternative approach to this problem consists in the infinite summation of the mutual impedance between one antenna and each of the infinite number of images of the second antenna. This summation converges very slowly and can be evaluated only by a modern digital computer. However, certain approximations may be made if the guard plane separation and dipole separation are both small, in which case it is seen that the mutual reactance is less than the input reactance of either dipole alone by sixty times the dipole separation in radians. This establishes the initial slope of the curves in Fig. 3 and also shows that, for the assumptions indicated, the change in the mutual reactance as the dipole separation changes is primarily due to the change in mutual reactance between antenna #2 and the basic element of the infinite array. In other words, the mutual reactance between antenna #2 and the remainder of the infinite array is essentially constant.

As one of the two antennas is moved toward either of the guard planes from its central location, the other antenna remaining fixed, the mutual reactance decreases, being almost linear for a small original separation and almost cosinusoidal for a large original separation. The initial slope of the linear decrease is the same as that of Fig. 3, and it is therefore evident that for small separations between antenna #2 and antenna #1, the mutual reactance is essentially independent of the direction of the separation.

Thick Input Impedance

We consider here the input impedance of a dipole of length L, centered between two parallel planes, and having a circular cross section with radius r_o or a strip cross section of width 2b and zero thickness. The basic problem as it refers to a circular cross section is shown in Fig. 5(a).

The problem is more easily considered physically if we replace the parallel planes by an infinite array of identical dipoles, fed with currents alternating in phase, as suggested by Fig. 5(b). The input impedance of the center antenna when in free space, or with the others open-circuited, is assumed known. We therefore must find the mutual coupling existing between the central element and the infinite array. This coupling is essentially independent of the radii of the elements of the array, as long as a small ratio of radius to spacing is maintained. We therefore find the mutual coupling between a thick central conductor and an infinite array of filamentary dipoles, outlined by Fig. 5(c). If this coupling is defined as Z_m, then $Z_{in} = Z_m + Z_{in0}$, where Z_{in} is the input impedance of the thick antenna between parallel planes and Z_{in0} is the input impedance of the same antenna in free space.

In order to find Z_m it is necessary to consider the thick central element as composed of differential filaments, and to find the mutual coupling, Z_m', between one such differential filament and the infinite array of filamentary dipoles. This step is show as Fig. 6(a). Z_m is then the average of Z_m' over the surface of the thick antenna, or, in other words, Z_m is equal to Z_m' evaluated at the average differential filament.

In the case of a circular cross section, it was shown above that the mutual impedance between two filamentary dipoles between parallel planes is sensitive only to the magnitude of their relative displacement and not to the direction. We therefore take $Z_m = Z_m'$, evaluating Z_m' for the filament indicated in Fig. 6(b). Fig. 6(c) applies to the strip.

Finally, we find Z_m' by replacing the central element, as in Fig. 6(d), obtaining $Z_m' = Z_{12} - Z_{120}$, where Z_{12} is the mutual impedance between the average differential filament and the infinite array, including this replaced central element and Z_{120} is the mutual impedance between the average differential filament and the replaced central element in free space.

In terms of reactances,

$$X_{in} = X_{12} - X_{120} + X_{in0} \quad (6)$$

where

- X_{in} = input reactance of thick antenna between parallel planes
- X_{in0} = input reactance of thick antenna in free space,
- X_{120} = mutual reactance between average differential filament of thick antenna and replaced central filamentary dipole in free space
- X_{12} = mutual reactance between average differential filament of thick antenna and replaced central filamentary dipole, both between parallel planes.

The methods of the previous section allow the approximate evaluation of X_{12} for dipoles of full length greater than 160°.

We obtain,

$$X_{12} \doteq (2+\cos\beta L)X_{12r} - 60 \sin\beta L \ \ln \frac{1+e^{-\pi y/H}}{1-e^{-\pi y/H}} \quad (7)$$

where X_{12r} applies to resonant lengths and is given by Eq. (5).

Using expressions available[3] for X_{in0} and X_{120}, we then have,

$$X_{in} \doteq (1 + \cos\beta L)X_{12r} - 60 \sin\beta L \; \ell n \frac{\sqrt{2}\,H}{\pi\gamma} \qquad (8)$$

For the circular cross section of radius r_o, we see that $\gamma = 2\pi r_o/\lambda$ and in the case of the strip of width 2b, $\gamma = \pi b/\lambda$.

Fig. 7 shows the variation of reactance, referred to the current loop, for several values of γ and H. With one exception, the reactance is shown only for dipole lengths greater than 160° in order to avoid excessive error. The curve for γ=.C, H $=\pi/4$ has been extended to $\beta L = 60°$ by graphical integration methods in order to show the asymptotic nature of these curves with those for a dipole in free space for short dipole lengths. It is not surprising that the reactance of a short dipole between parallel planes and the same dipole in free space should approach each other as the dipole length decreases, for, if we consider the infinite array of images, it is evident that the coupling from the central antenna to the images is becoming very small and the main contribution must be the input impedance of the central element itself.

References

1. W. Magnus and F. Oberhettinger, "Formeln und Sätze für die Speziellen Funktionen der Mathematischen Physik," p. 62, Springer, Berlin; 1948.
2. A more complete and rigorous discussion of the subject covered by this paper may be found in the author's Ph.D. thesis, "A Study of Radiation and Mutual Impedance Problems in Strip Transmission Lines," Univ. of Ill.; 1954.
3. E. C. Jordan, "Electromagnetic Waves and Radiating Systems," Prentice-Hall, pp. 352-363 (as corrected); 1950.

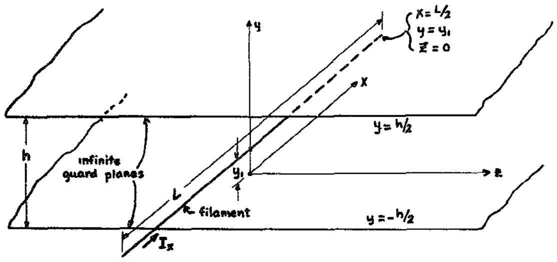

Fig. 1 - Filamentary dipole between parallel planes.

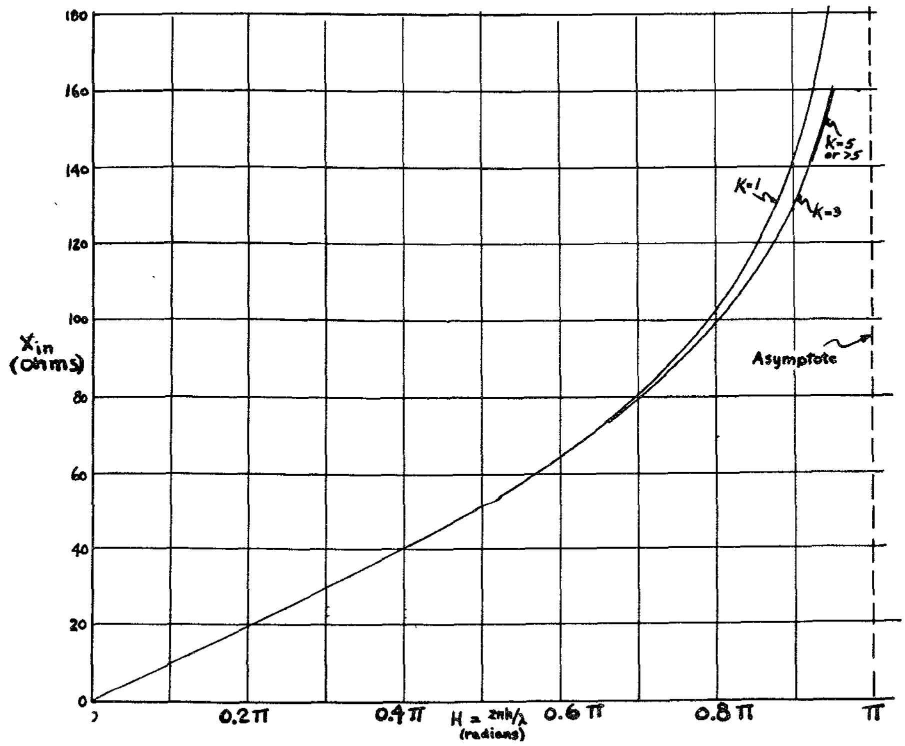

Fig. 2 - Input reactance of resonant filamentary dipole between parallel planes.

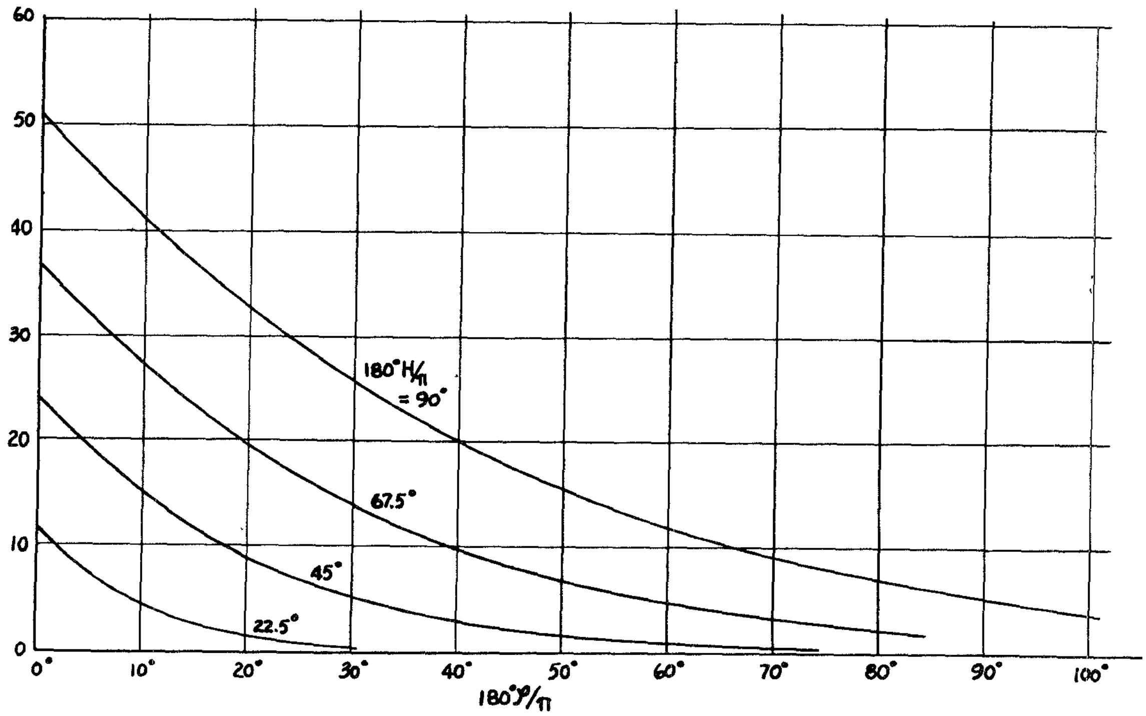

Fig. 3 - Mutual reactance between two parallel half-wave dipoles between parallel planes as a function of dipole separation.

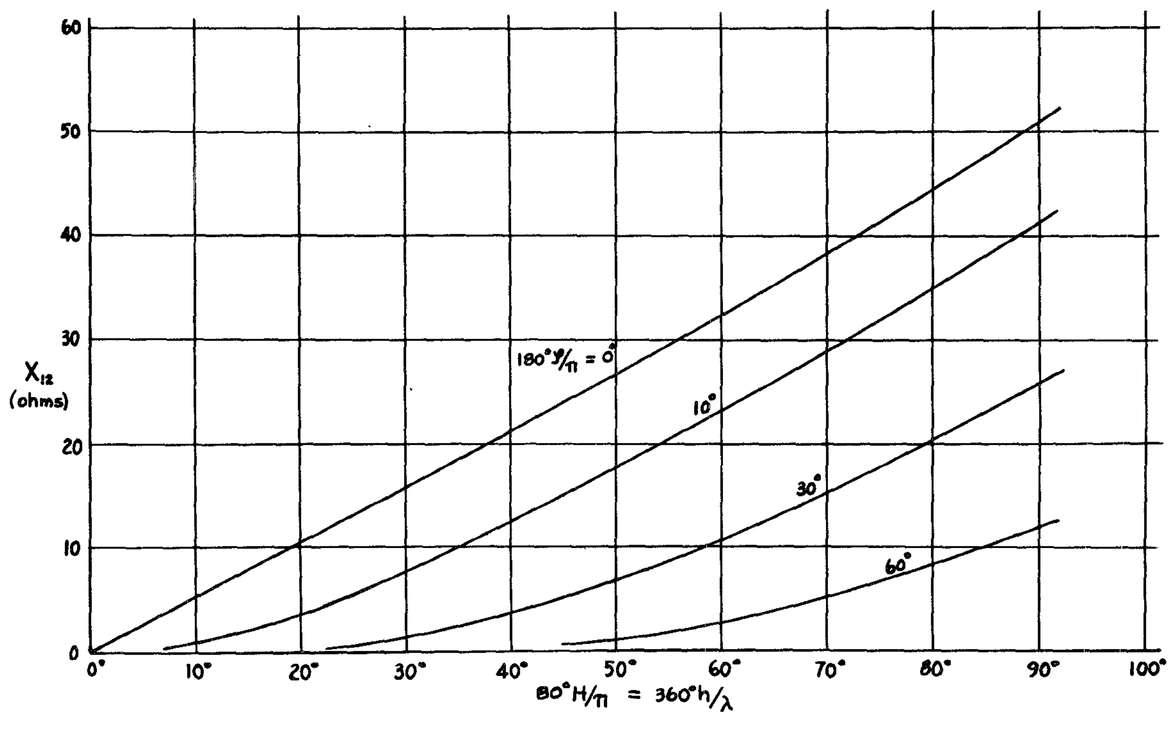

Fig. 4 - Mutual reactance between two parallel half-wave dipoles between parallel planes as a function of plane separation.

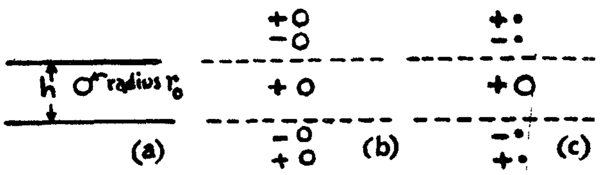

Fig. 5(a) - Dipole of circular cross-section between parallel planes.
(b) - Image representation.
(c) - Approximation obtained by replacing all images by filamentary dipoles.

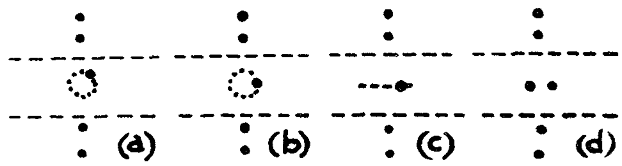

Fig. 6(a) - The infinite array of filamentary images and a representative differential filament.
(b) - The location of the differential filament for small radius.
(c) - The location of the differential filament for small strip width.
(d) - The infinite array of filamentary images, the special differential filament of (b) and (c), and the replaced central filament.

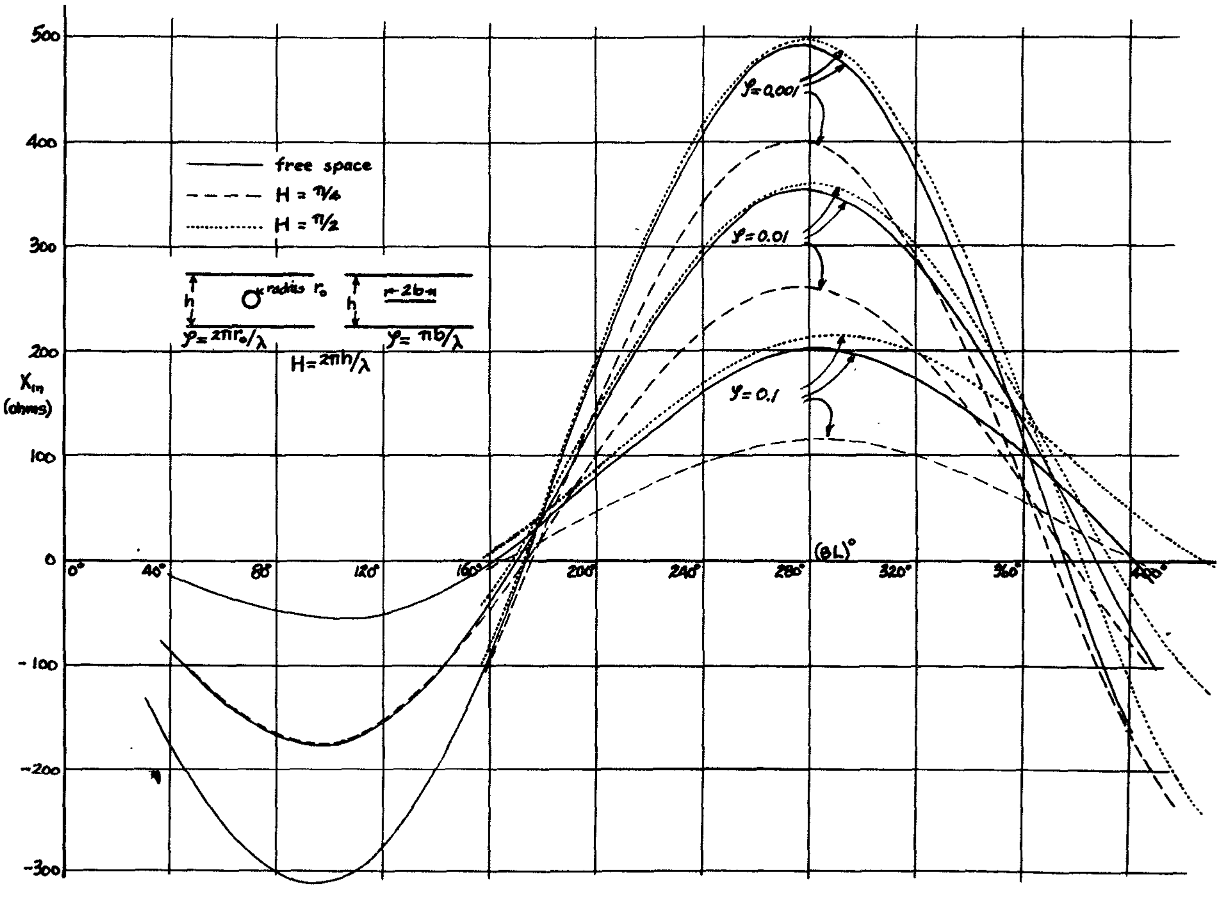

Fig. 7 - Input reactance of thick dipole between parallel planes. (NOTE: Reactance is referred to current loop. Reactance referred to dipole center is $1/\sin^2\frac{1}{2}\beta L$ times this value.)